MATH 2020 A
Fall 2020.

Lecture 21

- Stokes' theorem

Green's theorem states that

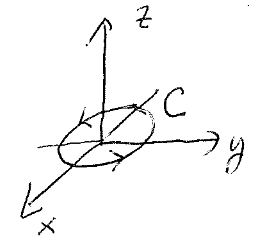
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy.$$

Stokes' theorem is the curved version of Green's Theorem where the plane region D is replaced by a surface S in $\mathbb{R}^3$ and the curve C is replaced by a curve C in space.

To guess what Stokes' thm would look like, consider

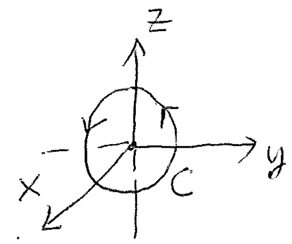
$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}.$$

When C is a circle lying on the x-y plane, one would get the formula above.



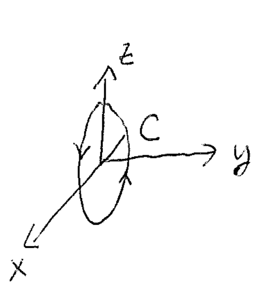
When C is a circle on the y-z plane, we get

$$\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dA = \int_C Q\,dy + R\,dz$$



When C is a circle on the z-x plane, we get

$$\iint_S \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dA = \int_C P\,dx + R\,dz.$$



therefore, for a general S lying in space, the RHS of Stokes' should contain

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}.$$

Let $\vec{F} = (P, Q, R)$ or $P\hat{i} + Q\hat{j} + R\hat{k}$ a C^1-v.f. in $\mathbb{R}^3$. Define its curl to be a v.f.

$$\operatorname{curl} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(-\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.$$

Let $\nabla = (\partial_x, \partial_y, \partial_z)$ be the "gradient operator". The curl vector could be computed via

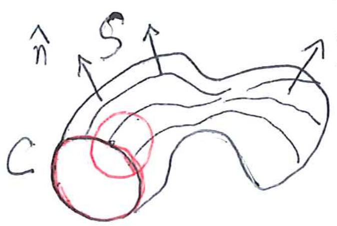
$$\det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}.$$

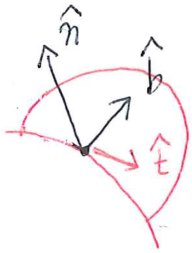
The notation $\nabla \times \vec{F}$ is often used instead of curl $\vec{F}$.

<u>Theorem 1</u> (<u>Stokes' thm</u>) <u>Let</u> S <u>be an oriented surface</u> in $\mathbb{R}^3$ <u>whose boundary is a</u> C^1 <u>simple, closed curve</u> C. <u>For a</u> C^1-v.f. $\vec{F}$ <u>on</u> S,

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \oint_C P\,dx + Q\,dy + R\,dz.$$

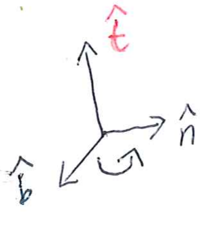
<u>Here</u> $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ <u>and</u> $\hat{n}$ <u>is the chosen unit normal on</u> S. <u>The orientation of</u> C <u>is determined by</u> $\hat{n}$ <u>according to the following right hand rule</u>.





$\hat{b}$ = binormal, the tangent vector perpendicular to the boundary and pointing inward S.

$\hat{b} \to \hat{n} \mapsto \hat{t}$



If we take S lying on the x-y plane and $\hat{n} = \hat{k}$, the boundary C is on the xy-plane running in anticlockwise direction. So

$$\nabla \times \vec{F} \cdot \hat{n} = \nabla \times \vec{F} \cdot \hat{k}$$
$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Any parametrization of C is of the form $(x(t), y(t), 0)$ so $\vec{r}\,'(t) = (x'(t), y'(t), 0)$. Thus

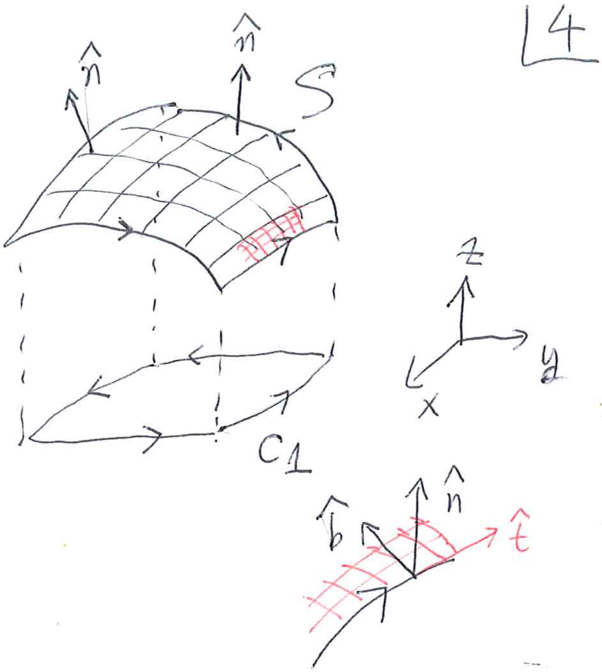
$$\oint_C P\,dx + Q\,dy + R\,dz$$
$$= \int_a^b \left(P(\vec{r}(t))\, x'(t) + Q(\vec{r}(t))\, y'(t) + R(\vec{r}(t))\, 0\right) dt$$
$$= \int_a^b P(\vec{r}(t))\, x'(t) + Q(\vec{r}(t))\, y'(t)\, dt$$
$$= \oint_C P\,dx + Q\,dy.$$

So,
$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$
$$\oint_C P\,dx + Q\,dy + R\,dz = \oint_C P\,dx + Q\,dy,$$
Stokes' thm reduces to Green's Theorem.

Proof of Stokes' thm. We give a proof in the special case where
$$S = \{ (x, y, f(x,y)) : (x,y) \in D \}$$
and D is bounded by a simple, closed curve C_1.

The standard parametrization of S

$$(x, y) \xrightarrow[\vec{r}]{} (x, y, f(x,y))$$



$$\vec{r}_x = (1, 0, f_x)$$

$$\vec{r}_y = (0, 1, f_y)$$

$$\vec{r}_x \times \vec{r}_y = (-f_x, -f_y, 1)$$

$$\hat{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+|\nabla f|^2}}$$

$$\therefore \iint_S \nabla\times\vec{F}\cdot\hat{n}\, d\sigma = \iint_S \nabla\times\vec{F}\cdot\frac{(-f_x, -f_y, 1)}{\sqrt{1+|\nabla f|^2}}\, d\sigma$$

$$= \iint_D \nabla\times\vec{F}\cdot(-f_x, -f_y, +1)\, dA(x,y)$$

$$= \iint_D \left[(R_y - Q_z)(-f_x) + (-R_x + P_z)(-f_y) + (Q_x - P_y)\right] dA(x,y) \quad \text{— } 1$$

On the other hand, let $\vec{\gamma}(t) = (x(t), y(t), f(x(t), y(t)))$ be a parametrization of C, the boundary of S, (so $t \mapsto (x(t), y(t))$ parametrizes C_1).

$$\oint_C P\,dx + Q\,dy + R\,z = \int_a^b \left[P(\vec{\gamma}(t))\,x'(t) + Q(\vec{\gamma}(t))\,y'(t) + R(\vec{\gamma}(t))\,z'(t)\right] dt$$

$$= \int_a^b P(\vec{\gamma}(t))\,x'(t) + Q(\vec{\gamma}(t))\,y'(t) + R(\gamma(t))(f_x x' + f_y y')\, dt$$

$$= \int_a^b \left[(P + Rf_x)\,x' + (Q + Rf_y)\,y'\right] dt$$

note that $P + Rf_x = P(\vec{\gamma}(t)) + R(\vec{\gamma}(t))\,f_x(x(t), y(t))$, etc.

Let $\tilde{P}(x,y) = P(x,y,f(x,y)) + R(x,y,f(x,y))\,f_x(x,y)$, &

$\tilde{Q}(x,y) = Q(x,y,f(x,y)) + R(x,y,f(x,y))\,f_y(x,y)$.

Then $\displaystyle\int_a^b [(P+Rf_x)x' + (Q+Rf_y)y']\,dt$

$$= \oint_{C_1} \tilde{P}\,dx + \tilde{Q}\,dy$$

(apply Green's to $\tilde{P}\hat{i} + \tilde{Q}\hat{j}$ on D)

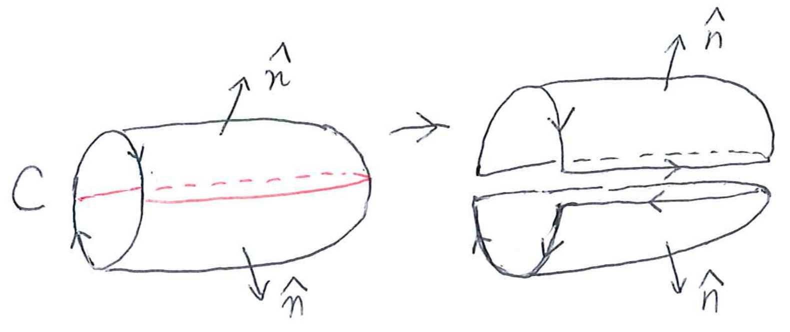
$$= \iint_D \left(\frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y}\right) dA$$

$$= \iint_D \Big[Q_x + Q_z f_x + (R_x + \cancel{R_z f_x})f_y + \cancel{R f_{yx}} - P_y - P_z f_y - (R_y + \cancel{R_z f_y})f_x - \cancel{R f_{xy}}\Big]\,dA$$

$$= \iint_D \Big[(Q_z - R_y)f_x + (R_x - P_z)f_x + Q_x - P_y\Big]\,dA$$

$= ①$, done.

As in the case of Green's thm, the general case can be proved by cutting up S.



$S \mapsto S_1, S_2$

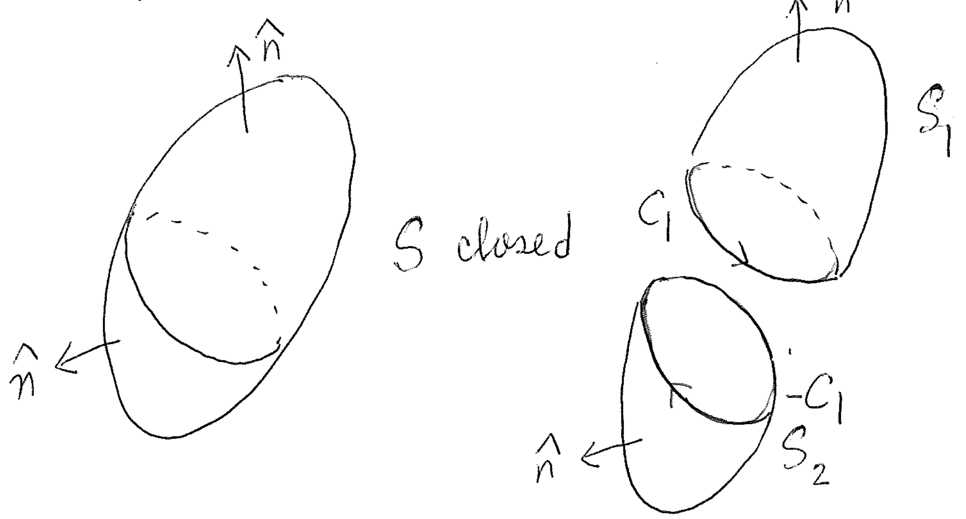
both S_1, S_2 are graphs over xy-plane.

Corollary 1

When S is a closed surface, that's, C is empty, Stokes' Theorem becomes

$$\oiint_S \nabla \times \vec{F} \cdot \hat{n} \, d\sigma = 0$$

Proof. Cut off into two ; to induce C and $-C$.



$$\oiint_S = \iint_{S_1} + \iint_{S_2}$$

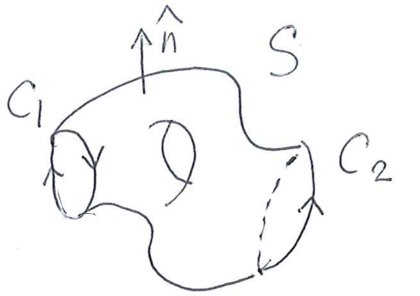
$$= \oint_{C_1} + \oint_{-C_1} = 0 .$$

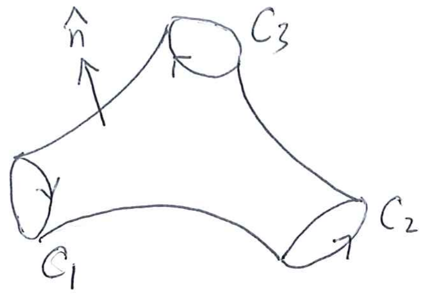
Corollary 2

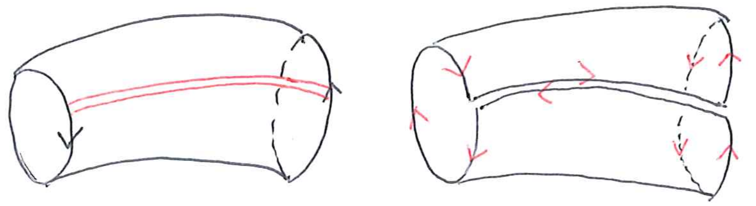
When S bounds several simple, closed curves $C_1, \ldots, C_n$,

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, d\sigma = \sum_{j=1}^{n} \oint_{C_j} \vec{F} \cdot d\vec{r} .$$

Pf: Cutting







a single long boundary C

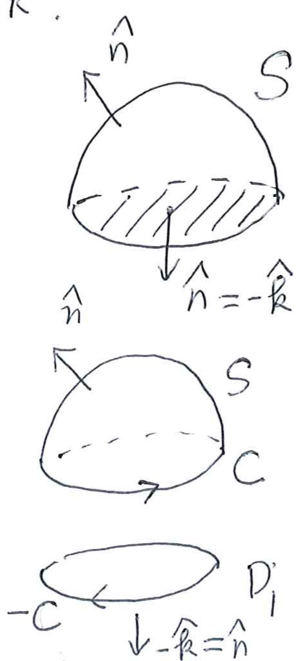
$\boxed{\text{e.g. } 1}$ Let S be the upper hemisphere $x^2+y^2+z^2=1,\ z \geq 0$, with normal pointing out. Evaluate

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma\ ,\quad \vec{F} = y\hat{i} + x^2z\hat{j} + e^{xz}\hat{k}.$$

We have

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & x^2z & e^{xz} \end{vmatrix}$$

$$= -x^2\hat{i} - ze^{xz}\hat{j} + (2xz-1)\hat{k}.$$



Let $S_1 = D_1$ the x-y plane unit disk sitting in $\mathbb{R}^3$ as a surface with normal pointing down (so that $S\cup S_1$ becomes a closed surface with outer normal.

By
$$\iint_{S\cup S_1} \nabla\times F\cdot\hat{n}\,d\sigma = 0,$$

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma = -\iint_{S_1} \nabla\times\vec{F}\cdot\hat{n}\,d\sigma$$

$$= -\iint_{S_1} \left(x^2\hat{\imath} - z e^{xz}\hat{\jmath} + (2xz-1)\hat{k}\right)\cdot(-\hat{k})\,d\sigma$$

$$= \iint_{S_1} (2xz-1)\,d\sigma$$

$d\sigma = \sqrt{1+|\nabla f|^2}\,dA$

Using the parametrizatin $(x, y) \mapsto (x, y, 0)$, $z = f(x,y) \equiv 0$.

$d\sigma = dA$

$$= \iint_{D_1} -dA$$

$$= -\pi \,.\ \#$$

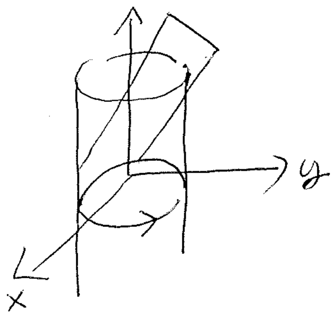
$\boxed{\text{e.g. 2}}$ Evaluate

$$\oint_C (y+z)dx + (z+x)dy + x\,dz$$ where

C is the intersection of $x^2+y^2=1$ (cylinder)

and $x+y+2z=10$ (plane).

the orientation is anticlockwise direction.



$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y+z & z+x & x \end{vmatrix}$$

$$= -\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = -\hat{\imath}.$$

$(x,y) \mapsto (x, y, \frac{1}{2}(10-x-y))$ parametrise the plane,

$$\hat{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+|\nabla f|^2}} = \frac{(\frac{1}{2}, \frac{1}{2}, 1)}{\sqrt{\frac{3}{2}}}$$ pointing up.

By Stokes' Thm

$$\oint_C (y+z)\,dx + (z+x)\,dy + x\,dz$$

$$= \iint_S -\hat{i}\cdot\hat{n}\,d\sigma = -\iint_S \frac{\frac{1}{2}}{\sqrt{\frac{3}{2}}}\,d\sigma \qquad d\sigma = \sqrt{1+|\nabla f|^2}\,dA = \sqrt{\tfrac{3}{2}}\,dA$$

$$= -\iint_{P_1} \frac{1}{2}\,dA$$

$$= -\frac{\pi}{2}. \quad \#.$$

Here is another application of Stokes' thm.

Recall we stated the following theorem:

<u>theorem</u> $(n=2)$ Let $\vec{F}$ be a C^1-v.f. in some open set $G \subseteq \mathbb{R}^2$.

$\vec{F} = P\hat{i} + Q\hat{j}$ has a potential

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

$(n=3)$ Let $\vec{F}$ be a C^1-v.f. in some open $G \subseteq \mathbb{R}^3$.

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ has a potential

$$\Rightarrow \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x},\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \text{ ie}$$

$$\nabla \times \vec{F} = \vec{0}.$$

Using Green's theorem, we have shown that the compatibility condition $P_y = Q_x$ is also sufficient when G is simply-connected.

<u>Theorem</u> $(n=2)$ As above, in addition G is simply connected.

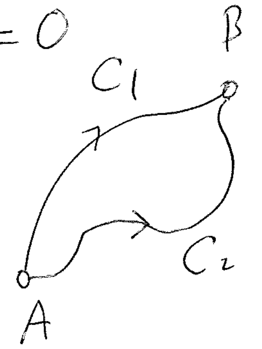
Then
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F} = P\hat{i} + Q\hat{j} \text{ has a potential.}$$

Now, we have

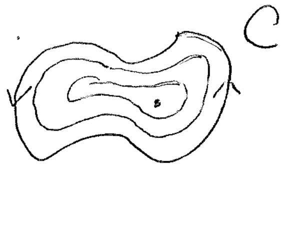
<u>Theorem</u> $(n=3)$ As above, in addition G is simply connected.

Then
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{or} \quad \nabla \times \vec{F} = \vec{0}.$$
$$\Rightarrow \vec{F} = P\hat{i} + Q\hat{j} + R\hat{k} \text{ has a potential.}$$

Pf: We first show that
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy + R\,dz = 0$$



for every simple, closed curve in G.
Using this property we can define the potential of $\vec{F}$, see



By assumption, G is simply connected. We can deform C onto a point. The deformation forms a surface whose boundary is C. Let the orientation of S be induced by the orientation of C. By Stokes' theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma.$$

However, one observes that the compatibility conditions are nothing but $\nabla\times\vec{F}=\vec{0}$. So

$$\oint_C \vec{F}\cdot d\vec{r} = 0, \qquad \text{done. } \#$$

So, finally we have shown that in a simply connected region, a vector field is conservative if and only if

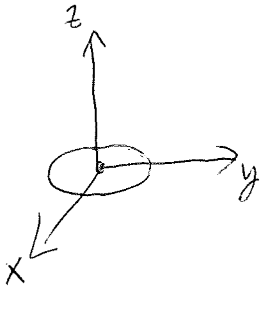
$$\nabla\times\vec{F}=\vec{0},$$

that is, it is curl free. We can express this result as

$$\text{curl } \vec{F}=0 \iff \exists\, \Phi \text{ s.t. } \vec{F}=\nabla\Phi.$$

Note that $\nabla\times\nabla\Phi=0$ (a conservative v.f. has no curl) no matter the region is simply connected or not.

Next, we examine the meaning of $\nabla\times\vec{F}$.



Let (x_0,y_0,z_0) be a point and use it as the origin. Consider the circle $C_\varepsilon=(\varepsilon\cos\theta, \varepsilon\sin\theta, 0)$ $\theta\in[0,2\pi]$.

Stokes theorem :

$$\oint_{C_\varepsilon} P\,dx + Q\,dy + R\,dz = \iint_{D_\varepsilon} \nabla\times\vec{F}\cdot\hat{k}\,d\sigma$$

$$\therefore \lim_{\varepsilon\to 0} \frac{1}{|D_\varepsilon|} \underbrace{\oint_{C_\varepsilon} P\,dx + Q\,dy + R\,dz}_{\text{circulation around } C_\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{|D_\varepsilon|} \iint_{D_\varepsilon} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(x_0, y_0, z_0).$$

So $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(x_0, y_0, z_0)$ measures the strength of the v.f. $\vec{F}$ rotates around the z-axis.

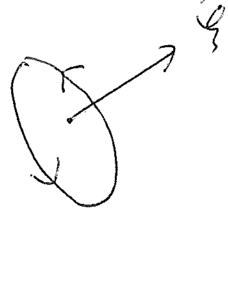
Similarly,

$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)(x_0, y_0, z_0)$ measures the strength of $\vec{F}$ at (x_0, y_0, z_0) when rotates around the x-axis.

$\left(-\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}\right)(x_0, y_0, z_0)$ measure ------

------------ y-axis.

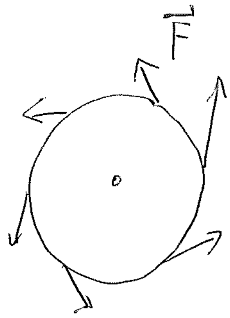
How about around the ξ-axis?



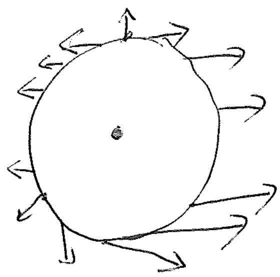
It is $\displaystyle\lim_{\varepsilon\to 0} \frac{1}{|D_\varepsilon|} \oint_{C_\varepsilon} P\,dx + Q\,dy + R\,dz$

$$= \lim_{\varepsilon \to 0} \frac{1}{|D_\varepsilon|} \iint_{D_\varepsilon} \nabla \times \vec{F} \cdot \hat{\varphi} \, d\sigma$$

$$= \nabla \times \vec{F} \cdot \hat{\varphi} \, . \quad \#$$



when $\vec{F}$ rotates around the pt, the curl at the pt > 0 or < 0.



$\vec{F}$ not nec. around the pt, curl $\vec{F}$ could still be non-zero.